On the Solvability of a Nonlinear Optimal Control Problem for the Thermal Processes Described by Fredholm Integro-Differential Equations

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Abstract The problem of nonlinear optimal control of the thermal process described by Fredholm integro-differential equation was investigated. The concept of a weak generalized solution of the boundary problem was introduced and the algorithm for its construction was indicated. It was established that the optimal control is defined as a solution of a nonlinear integral equation satisfying the additional condition in the form of inequality. Sufficient conditions for unique solvability of nonlinear optimization were found and the algorithm for constructing approximate solutions was developed. The convergence of approximate solutions with respect to control, optimal process and functional was investigated.

Keywords Boundary value problem · Weak generalized solution · Functional · The maximum principle · The optimality condition · Integral equation · Approximate solution · Convergence

1 Introduction

Many applied problems are described by integro-differential equations [5, 7]. As it was noticed in [3, Introduction] in many applications mathematical models which contain integro-differentional operators haven't been studied or have been studied not enough because of controlled system's difficulty. Problems of control processes described by integro-differential equations, in the case in which control functions enter the equations non-linearly, were almost not studied. In this paper, the solvability of control problem with the quadratic quality criterion was investigated. Using the maximum principle in the case in which the controlled process is described by a Fredholm integro-differential equation, the optimality condition was obtained in the form of a nonlinear integral equation and differential inequality, i.e. optimal control is defined as a solution of the specific problem that is new in the theory of integral equations. By applying the method of [2] sufficient conditions were found for unique solvability of this problem and the algorithm was indicated for constructing

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solutions of nonlinear optimization problems with arbitrary precision in the form of a triplet $(u^0(t), v^0(t,x), J[u^0(t)])$, where $u^0(t)$ is the optimal control, $v^0(t,x)$ is the optimal process, $J[u^0(t)]$ is the minimal value of the functional.

2 Boundary Value Problem of the Controlled Process

Let the state of a thermal process be described by a scalar function $v(t,x)$, which in the region $Q_T = Q \times (0,T]$, where Q is a region of the space R^n bounded by a piecewise smooth curve γ, satisfies the integral-differential equation [5, 7]

$$\begin{aligned} &v_t - Av = \lambda \int_0^T K(t,\tau)v(\tau,x)d\tau + g(t,x)f\big[t,u(t)\big], \\ &x \in Q \subset R_n,\ 0 < t \le T \end{aligned} \tag{2.1}$$

and on the boundary of Q satisfies the initial condition

$$v(0,x) = \psi(x), \quad x \in Q \tag{2.2}$$

and the boundary condition

$$\begin{aligned} &\Gamma v(t,x) \equiv \sum_{i,j=1}^{n} a_{ij}(x)v_{x_j}(t,x)\cos(\delta,x_i) + a(x)v(t,x) = 0, \\ &x \in q,\ 0 < t \le T. \end{aligned} \tag{2.3}$$

Here A is the elliptic operator defined by the formula:

$$Av(t,x) \equiv \sum_{i,j=1}^{n} \big(a_{ij}(x)v_{x_j}(t,x)\big)_{x_i} - c(x)v(t,x),$$

$$a_{ij}(x) = a_{ji}(x), \quad \sum_{i,j=1}^{n} a_{ij}(x)a_i a_j \ge a_0 \sum_{i=1}^{n} a_i^2, \quad a_0 > 0;$$

δ is a normal vector, outgoing from the point $x \in q$; T is a fixed moment of time, $K(t,\tau)$ is a given function defined in the region $D = (0 \le t \le 1, 0 \le \tau \le 1)$ and satisfying the condition

$$\int_0^T \int_0^T K^2(t,\tau)d\tau dt = K_0 < \infty, \tag{2.4}$$

i.e. $K(t,\tau)$ is an element of the Hilbert space $H(D) \equiv L_2(D)$;

$$\begin{aligned} &g(t,x) \in H(Q), \qquad \psi(x) \in H(0,1), \qquad f\big[t,u(t)\big] \in H(0,T), \\ &f_u\big[t,u(t)\big] \ne 0, \quad \forall t \in (0,T), \end{aligned} \tag{2.5}$$

are given functions; $a(x) \geq 0$, $c(x) \geq 0$ are known measurable functions; $u(t) \in H(0, T)$ is a control function, λ is a parameter and $\alpha > 0$ is a constant.

As is known, under conditions (2.5) problem (2.1)–(2.3) has no classical solutions. Therefore, we will use the notion of a weak generalized solution of problem (2.1)–(2.3).

The solution of problem (2.1)–(2.3) we will seek in the form:

$$\begin{aligned} v(t,x) &= \sum_{n=1}^{\infty} v_n(t) z_n(x), \\ v_n(t) &= \big(v(t,x), z_n(x)\big) = \int_Q v(t,x) z_n(x) dx, \end{aligned} \tag{2.6}$$

where $z_n(x)$, $n = 1, 2, 3, \ldots$ are eigenfunction function of the boundary value problem

$$Az(x) = -\lambda^2 z(x), \quad x \in Q,$$
$$\Gamma z(x) = 0, \quad x \in q,$$

which form a complete orthonormal system in the Hilbert space $H(Q)$, and the corresponding eigenvalues λ_n satisfy the following conditions

$$\lambda_n \leq \lambda_{n+1} \leq \cdots, \quad \lim_{n\to\infty} \lambda_n = \infty$$

Definition 2.1 A weak generalized solution of problem (2.1)–(2.3) is a function $v(t,x) \in H(Q)$ that satisfies the initial condition in a weak sense, i.e. for any function $\phi_0(x) \in H(Q)$ we have the equality:

$$\lim_{t\to+0} \int_Q v(t,x)\phi_0(x)dx = \int_Q \psi(x)\phi_0(x)dx,$$

and the Fourier coefficients $v_n(t)$ satisfy the linear Fredholm integral equation of the second type

$$\begin{aligned} v_n(t) = &\int_0^t e^{-\lambda_n^2(t-\tau)}\left(\lambda \int_0^T K(\tau,s)v_n(s)ds + g_n(\tau) f\big[\tau, u(\tau)\big]\right)d\tau \\ &+ e^{-\lambda_n^2 t}\psi_n, \end{aligned} \tag{2.7}$$

where ψ_n and $g_n(t)$ are the Fourier coefficients of the functions $\psi(x)$, $g(t,x)$ respectively.

To determine the Fourier coefficients $v_n(t)$ (2.7) can be rewritten as

$$v_n(t) = \lambda \int_0^T K_n(t,s)v_n(s)ds + \alpha_n(t), \tag{2.8}$$

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where

$$K_n(t,s)=\int_0^t e^{-\lambda_n^2(t-\tau)}K(\tau,s)d\tau, \tag{2.9}$$

$$\alpha_n(t)=e^{-\lambda_n^2 t}\psi_n+\int_0^t e^{-\lambda_n^2(t-\tau)}g_n(\tau)f\big[\tau,u(\tau)\big]d\tau \tag{2.10}$$

The solution of integral equation (2.8) we find by the formula [1]:

$$v_n(t)=\lambda\int_0^T R_n(t,s,\lambda)\alpha_n(s)ds+\alpha_n(t), \tag{2.11}$$

where

$$R_n(t,s,\lambda)=\sum_{n=1}^{\infty}\lambda^{i-1}K_{n,i}(t,s),\quad n=1,2,3,\ldots, \tag{2.12}$$

is the resolvent $K_{n,1}(t,s)\equiv K_n(t,s)$, and the iterated kernels $K_{n,i}(t,s)$ are defined by the formula [1]

$$K_{n,i+1}(t,s)=\int_0^T K_n(t,\eta)K_{n,i}(\eta,s)d\eta,\quad i=1,2,3,\ldots, \tag{2.13}$$

for each $n=1,2,3,\ldots$. We investigate the convergence of Neumann series (2.12). According to (2.9) and (2.13) by direct calculation the following estimates are established

$$\big|K_{n,i}(t,s)\big|^2\le\frac{(K_0T)^{i-1}}{(2\lambda_n^2)^i}\int_0^T K^2(\eta,s)d\eta,\quad i=1,2,3,\ldots. \tag{2.14}$$

Neumann series (2.12) is dominated by the numerical series

$$\begin{aligned}\sum_{i=1}^{\infty}\lambda^{i-1}K_{n,i}(t,s)&\le\sum_{i=1}^{\infty}|\lambda|^{i-1}\big|K_{n,i}(t,s)\big|\\&\le\left(\int_0^T K^2(\eta,s)d\eta\right)^{1/2}\frac{1}{\sqrt{2\lambda_n^2}}\sum_{i=1}^{\infty}\left(|\lambda|\frac{\sqrt{K_0T}}{\sqrt{2\lambda_n^2}}\right)^{i-1},\end{aligned}$$

which converges for every $n=1,2,3,\ldots$ for the values of the parameter λ that satisfy the inequality

$$|\lambda|\frac{\sqrt{K_0T}}{\sqrt{2\lambda_n^2}}<1.$$

Note that

$$|\lambda|<\frac{\sqrt{2}}{\sqrt{K_0T}}\lambda_n\xrightarrow[n\to\infty]{}\infty,$$

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i.e. the radius of convergence increases when n is growing. However, the Neumann series, for the parameter values λ that satisfy the condition

$$|\lambda| < \frac{\sqrt{2}}{\sqrt{K_0 T}}\lambda_1, \quad \lambda \neq 0 \tag{2.15}$$

converges absolutely for any $n = 1, 2, 3, \ldots$. In this case the resolvent as the sum of an absolutely convergent series is a continuous function and satisfies the following estimates

$$\begin{aligned} \left|R_n(t,s,\lambda)\right| &\le \left(\int_0^T K^2(\eta,s)d\eta\right)^{1/2} \frac{1}{\sqrt{2\lambda_n^2}} \sum_{i=1}^{\infty}\left(|\lambda|\frac{\sqrt{K_0T}}{\sqrt{2\lambda_n^2}}\right)^{i-1} \\ &= \frac{1}{\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0T}}\left(\int_0^T K^2(\eta,s)d\eta\right)^{1/2}, \\ \int_0^T R_n^2(t,s,\lambda)ds &\le \frac{1}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0T})^2}\int_0^T\int_0^T K^2(\eta,s)d\eta ds \\ &= \frac{K_0}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0T})^2}. \end{aligned} \tag{2.16}$$

Thus, the solution of problem (2.1)–(2.3) we find by (2.6), where $v_n(t)$ is defined by formula (2.11) as the unique solution of integral equation (2.8). It is easy to verify that this solution satisfies initial condition (2.2).

Now we show that this solution is an element of the space $H(Q_T)$. Taking into account (2.9) and (2.10) by direct calculation it is easy to show that the following inequality holds

$$\begin{aligned} &\int_0^T\int_Q v^2(t,x)dxdt \\ &\le \int_0^T\int_Q\left(\sum_{n=1}^{\infty} v(t)z_n(x)\right)^2 dxdt = \int_0^T\sum_{n=1}^{\infty} v_n^2(t)dt \\ &\le \int_0^T\sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)\alpha_n(s)ds + \alpha_n(t)\right)^2 dt \\ &\le 2\int_0^T\sum_{n=1}^{\infty}\left(\lambda^2\int_0^T R_n^2(t,s,\lambda)ds\int_0^T \alpha_n^2(s)ds + \alpha_n^2(t)\right)dt \\ &\le 2\left(\frac{\lambda^2 K_0 T}{(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0T})^2}\sum_{n=1}^{\infty}\int_0^T \alpha_n^2(s)ds + \int_0^T\sum_{n=1}^{\infty}\alpha_n^2(t)dt\right) \end{aligned}$$

$$
\begin{aligned}
&\le 2\left(\frac{2\lambda^2 K_0 T}{(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T})^2}+1\right)\\
&\quad\times 2T\left(\sum_{n=1}^{\infty}\psi_n^2+\sum_{n=1}^{\infty}\int_0^T g_n^2(\tau)d\tau\int_0^T f^2\big[\tau,u(\tau)\big]d\tau\right)\\
&=4T\left(\frac{\lambda^2 K_0 T}{(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T})^2}+1\right)\\
&\quad\times\big\{\big\|\psi(x)\big\|_H^2+\big(\big\|g(t,x)\big\|_H^2\big\|f\big[t,u(t)\big]\big\|_H^2\big)\big\}.
\end{aligned}
$$

From this inequality it follows that $v(t,x)\in H(Q_T)$. When the functions $v_n(t)$, $n=1,2,3,\dots$, are determined by formulas (2.11)–(2.12), it is not always possible to find the exact resolvent $R_n(t,s,\lambda)$. In practice, the approximations of the resolvent are considered most often. The truncated series of the form

$$
R_n^m(t,s,\lambda)=\sum_{i=1}^{m}\lambda^{i-1}K_{n,i}(t,s),\quad n=1,2,3,\dots, \tag{2.17}
$$

is called mth approximation of the resolvent $R_n(t,s,\lambda)$ for each fixed $n=1,2,3,\dots$.

The function $v_n^m(t)$ defined by the formula

$$
v_n^m(t)=\lambda\int_0^T R_n^m(t,s,\lambda)\alpha_n(s)ds+\alpha_n(t),\quad n=1,2,3,\dots, \tag{2.18}
$$

is called the mth approximation of the function $v_n(t)$ for each fixed $n=1,2,3,\dots$.

According to the formula (2.6), the mth approximation of the solution $v(t,x)$ of boundary value problem (2.1)–(2.3) we find from the formula

$$
v^{(m)}(t,x)=\sum_{n=1}^{\infty}v_n^m(t)z_n(x), \tag{2.19}
$$

where $v_n^m(t)$ have the form (2.18). We show that the approximate solution $v_n^m(t,x)$ of boundary value problem (2.1)–(2.3) converges to the exact solution $v(t,x)$ with respect to the norm of the space $H(Q_T)$. Taking into account (2.12), (2.14), (2.15), (2.17), (2.18) and the inequality

$$
\begin{aligned}
&\sum_{i=m+1}^{\infty}\alpha^i\le\alpha^{m+1}+\int_{m+1}^{\infty}\alpha^x dx=\alpha^{m+1}+\frac{1}{\ln\alpha}\alpha^x\Big|_{m+1}^{\infty}=\alpha^{m+1}\left(1-\frac{1}{\ln\alpha}\right),\\
&\quad 0<\alpha<1,
\end{aligned}
$$

by direct computation we find that

$$\begin{aligned}
\left[v_n(t) - v_n^m(t)\right]^2 &= \left(\lambda \int_0^T \left[R_n(t,s,\lambda) - R_n^m(t,s,\lambda)\right]\alpha_n(s)ds\right)^2 \\
&\le \lambda^2 \int_0^T \left[R_n(t,s,\lambda) - R_n^m(t,s,\lambda)\right]^2 ds \int_0^T \alpha_n^2(s)ds \\
&\le \lambda^2 \int_0^T \left(\sum_{i=m+1}^{\infty} |\lambda|^{i-1}\left|K_{n,i}(t,s)\right|\right)^2 ds \int_0^T \alpha_n^2(s)ds \\
&\le \lambda^2 \frac{K_0}{2\lambda_n^2}\left(\sum_{i=m+1}^{\infty}\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{i-1}\right)^2 \int_0^T \alpha_n^2(s)ds \\
&\le \frac{\lambda^2K_0}{2\lambda_n^2}\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m}\left(1-\frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}\right)^2 \\
&\quad \times \int_0^T \alpha_n^2(s)ds \le C_n(\lambda)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m}, \qquad (2.20)
\end{aligned}$$

where

$$\begin{aligned}
C_n(\lambda) &= \frac{\lambda^2K_0}{2}\left(1-\frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}\right)^2 \\
&\quad \times \left(\psi_n^2(x) + \int_0^T g_n^2(\tau)d\tau \left\|f\left[t,u(t)\right]\right\|_H^2\right). \qquad (2.21)
\end{aligned}$$

Note that, because the parameter λ satisfies (2.15), we have the inequality

$$0 < 1 - \frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}} < \infty. \qquad (2.22)$$

The convergence of the approximate solutions of boundary-value problem follows from

$$\begin{aligned}
&\left\|v(t,x) - v^m(t,x)\right\|_H^2 \\
&= \int_0^T \int_Q \left(\sum_{n=1}^{\infty}\left[v_n(t) - v_n^m(t)\right]z_n(x)\right)^2 dxdt \\
&= \int_0^T \sum_{n=1}^{\infty}\left[v_n(t) - v_n^m(t)\right]^2 dt \le \int_0^T \sum_{n=1}^{\infty} C_n(\lambda)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m} dt
\end{aligned}$$

$$\leq \int_0^T \sum_{n=1}^{\infty} \frac{\lambda^2 K_0}{\lambda_1^2} \left(1 - \frac{1}{\ln|\lambda|\sqrt{\frac{K_0 T}{2\lambda_1^2}}}\right)^2$$

$$\times \left(\psi_n^2 + \int_0^T g_n^2(\tau)d\tau \left\| f\left[t, u(t)\right] \right\|_n^2\right) \left(|\lambda| \sqrt{\frac{K_0 T}{2\lambda_1^2}}\right)^{2m} dt$$

$$\leq \frac{\lambda^2 K_0 T}{\lambda_1^2} \left(1 - \frac{1}{\ln|\lambda|\sqrt{\frac{K_0 T}{2\lambda_1^2}}}\right)^2$$

$$\times \left(\left\|\psi(x)\right\|_H^2 + \left\|g(t,x)\right\|_H^2 \left\| f\left[t, u(t)\right] \right\|_H^2\right) \left(|\lambda| \sqrt{\frac{K_0 T}{2\lambda_1^2}}\right)^{2m} \xrightarrow[m \to \infty]{} 0.$$

3 Formulation of Optimal Control Problem and Conditions of Optimality

Consider the optimization problem in which it is required to minimize the integral functional

$$J\left[u(t)\right] = \int_Q \left[v(T,x) - \xi(x)\right]^2 dx + 2\beta \int_0^T M\left[t, u(t)\right] dt, \quad \beta > 0, \tag{3.1}$$

where $\xi(x) \in H(Q)$, $M_u[t, u(t)] \in H(0,T)$—are given functions on the set of solutions of problem (2.1)–(2.3), i.e. we need to find the control $u^0(t) \in H(0,T)$ which together with the corresponding solution $v^0(t,x)$ of boundary value problem (2.1)–(2.3) gives the smallest possible value of functional (3.1). In this case $u^0(t)$ is called the optimal control, and $v^0(t,x)$ the optimal process.

Since by condition (2.5) each control $u(t)$ uniquely defines the controlled process $v(t,x)$, the solution of boundary value problem (2.1)–(2.3) of the form $v(t,x) + \Delta v(t,x)$ corresponds to the control $u(t) + \Delta u(t)$, where $\Delta v(t,x)$ is the increment corresponding to the increment $\Delta u(t)$. According to the procedure of application of the maximum principle [3, 4, 6], the increment of the functional (3.1) can be written as

$$\Delta J[u] = J[u + \Delta u] - J[u]$$

$$= -\int_0^T \Delta \Pi\left[t, v(t,x), \omega(t,x), u(t)\right] dt + \int_Q \Delta v^2(T,x) dx, \tag{3.2}$$

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where

$$\Delta\Pi(t,v,\omega,u)=\Pi\big(t,v(t,x),\omega(t,x),u(t)+\Delta u(t)\big)-\Pi\big(t,v(t,x),\omega(t,x),u(t)\big),$$

$$\Pi\big(t,v(t,x),\omega(t,x),u(t)\big)=\int_Q g(t,x)\omega(t,x)dxf\big[t,u(t)\big]-2\beta M\big[t,u(t)\big], \tag{3.3}$$

and the function $\omega(t,x)$ is a solution of the adjoint boundary value problem

$$\begin{aligned}\omega_t+A\omega+\int_0^T K(\tau,t)\omega(\tau,x)d\tau&=0, \quad x\in Q,\ 0\le t<T,\\ \omega(T,x)+2\big[v(T,x)-\xi(x)\big]&=0, \quad x\in Q,\\ \Gamma\omega(t,x)&=0, \quad x\in q.\end{aligned} \tag{3.4}$$

According to the maximum principle for systems with distributed parameters [6], the optimal control is determined by the relations

$$2\beta M_u\big[t,u(t)\big]f_u^{-1}\big[t,u(t)\big]=\int_Q g(t,x)\omega(t,x)dx, \tag{3.5}$$

$$f_u\big[t,u(t)\big]\left(\frac{M_u[t,u(t)]}{f_u[t,u(t)]}\right)_u>0, \tag{3.6}$$

which are called the optimality conditions.

4 Solution of the Adjoint Boundary-Value Problem

We are looking for solution of boundary value problem (3.4) in the form of the series

$$\omega(t,x)=\sum_{i=1}^{\infty}\omega_n(t)z_n(x). \tag{4.1}$$

It is easy to verify that the Fourier coefficients $\omega(t,x)$ for each fixed $n=1,2,3,\ldots,$ satisfy the conditions

$$\begin{aligned}\omega_n'(t)-\lambda_n^2\omega_n(t)&=-\lambda\int_0^T K(\tau,t)\omega_n(\tau)d\tau,\\ \omega_n(T)+2\big[v_n(T)-\xi_n\big]&=0,\end{aligned}$$

which can be converted to the linear non-homogeneous Fredholm integral equation of the second type

$$\omega_n(t)=\lambda\int_0^T B_n(s,t)\omega_n(s)ds-2e^{-\lambda_n^2(T-t)}\big[v_n(T)-\xi_n\big], \tag{4.2}$$

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where the kernel

$$B_n(s,t)=\int_t^T e^{-\lambda_n^2(T-t)}K(s,\tau)d\tau \quad \text{and} \quad B_n(s,T)=0. \tag{4.3}$$

The solution of (4.2) we find by the formula [1]

$$\omega_n(t)=-2\big[v_n(T)-\xi_n\big]\bigg(e^{-\lambda_n^2(T-t)}+\lambda\int_0^T P_n(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds\bigg), \tag{4.4}$$

where the resolvent $P_n(s,t,\lambda)$ of the kernel $B_n(s,t)$ is given by

$$P_n(s,t,\lambda)=\sum_{i=1}^{\infty}\lambda^{i-1}B_{n,i}(s,t),$$

$$B_{n,i+1}(s,t)=\int_0^T B_n(\eta,t)B_{n,i}(s,\eta)d\eta, \quad i=1,2,3,\dots,$$

and by the condition (2.14) it is a continuous function, and satisfies the inequality

$$\big|P_n(s,t,\lambda)\big|\le\frac{1}{\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0T}}\bigg(\int_0^T K^2(\eta,s)d\eta\bigg)^{1/2}. \tag{4.5}$$

It is easy to verify that $\omega(t,x)$ is an element of the space $H(Q)$.
This follows from the inequality

$$\begin{aligned}
&\int_0^T\int_Q \omega^2(t,x)dxdt\\
&=\int_0^T\int_Q\bigg(\sum_{n=1}^{\infty}\omega_n(t)z_n(x)\bigg)^2dxdt=\int_0^T\sum_{n=1}^{\infty}\omega_n^2(t)dt\\
&\le 8\int_0^T\sum_{n=1}^{\infty}\big[v_n(T)-\xi_n\big]^2\\
&\quad\times\bigg(e^{-2\lambda_n^2(T-t)}+\lambda^2\int_0^T P_n^2(s,t,\lambda)ds\int_0^T e^{-2\lambda_n^2(T-s)}ds\bigg)dt\\
&\le 8\int_0^T\sum_{n=1}^{\infty}\big[v_n(T)-\xi_n\big]^2\\
&\quad\times\bigg(1+\lambda^2\frac{1}{(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0T})^2}\int_0^T\int_0^T K^2(s,\eta)d\eta ds\frac{1}{2\lambda_1^n}\bigg)dt
\end{aligned}$$

$$\leq 16T\left(1+\frac{\lambda^2 K_0}{(2\lambda_1^2\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T})^2}\right)\sum_{n=1}^{\infty}\left(v_n^2(T)-\xi_n^2\right)<\infty,$$

which holds by the following relations

$$\sum_{n=1}^{\infty} v_n^2(T)<\infty \quad \text{and} \quad \sum_{n=1}^{\infty}\xi_n^2=\left\|\xi(x)\right\|_H^2.$$

5 Nonlinear Integral Equation of Optimal Control

We find the optimal control according to optimality conditions (3.5) and (3.6). We substitute in (3.5) the solution of adjoint boundary value problem (3.4) defined by (4.1) and (4.4). First, we calculate the integral

$$\int_Q g(t,x)\omega(t,x)dx=\int_Q\sum_{n=1}^{\infty}g_n(t)z_n(x)\sum_{k=1}^{\infty}\omega_k(t)z_k(x)dx=\sum_{n=1}^{\infty}g_n(t)\omega_n(t)$$

and rewrite equality (3.5) in the form

$$\begin{aligned}&\beta M_u\left[t,u(t)\right]f_u^{-1}\left[t,u(t)\right]\\&\quad=-\sum_{n=1}^{\infty}g_n(t)\left[v_n(T)-\xi_n\right]\left(e^{-\lambda_n^2(T-t)}+\lambda\int_0^T P_n(s,t,\lambda)e^{-\lambda_n^2(T-s)}ds\right).\end{aligned}$$

According to (2.6), we reduce this equality to the form

$$\begin{aligned}&\beta M_u\left[t,u(t)\right]f_u^{-1}\left[t,u(t)\right]+\sum_{n=1}^{\infty}L_n(t,\lambda)\int_0^T G_n(s,\lambda)f\left[s,u(s)\right]ds\\&\quad=\sum_{n=1}^{\infty}L_n(t,\lambda)h_n,\end{aligned}\tag{5.1}$$

where

$$L_n(t,\lambda)=g_n(t)\left[e^{-\lambda_n^2(T-t)}+\lambda\int_0^T P_n(\tau,t,\lambda)e^{-\lambda_n^2(T-\tau)}d\tau\right],\tag{5.2}$$

$$G_n(t,\lambda)=g_n(t)\left[e^{-\lambda_n^2(T-t)}+\lambda\int_t^T R_n(T,\tau,\lambda)e^{-\lambda_n^2(\tau-t)}d\tau\right],\tag{5.3}$$

$$h_n=\xi_n-\psi_n\left[e^{-\lambda_n^2 T}+\lambda\int_0^T R_n(T,\tau,\lambda)e^{-\lambda_n^2\tau}d\tau\right].\tag{5.4}$$

Thus, the optimal control is defined as the solution of nonlinear integral equation (5.1), and here we must have condition (3.6). Condition (3.6) restricts the class of functions of external actions $f[t, u(t)]$. Therefore, we assume that the function $f[t, u(t)]$ satisfies (3.6) for any control $u(t) \in H(0, T)$.

Nonlinear integral equation (5.1) is solved according to the procedure of work [7]. We set

$$\beta M_u\big[t, u(t)\big] f_u^{-1}\big[t, u(t)\big] = p(t). \tag{5.5}$$

Lemma 5.1 *The function $p(t)$ is an element of space $H(0, T)$.*

Proof By (2.5), we have the estimate

$$\big|f_u^{-1}\big[t, u(t)\big]\big| \leq M_0, \quad \forall t \in [0, T]. \qquad \square$$

Since $u(t) \in H(0, T)$, the statement of the lemma follows by the inequality

$$\int_0^T p^2(t)dt \leq \beta^2 \int_0^T \big|f_u^{-1}\big[t, u(t)\big]\big|^2 \big|M_u\big[t, u(t)\big]\big|^2 dt$$
$$\leq \beta^2 M_0^2 \int_0^T M_u^2\big[t, u(t)\big](t)dt < \infty.$$

According to (3.6), the control $u(t)$ is uniquely determined by equality (5.5), i.e. there is a function φ such that

$$u(t) = \varphi\big(t, p(t), \beta\big). \tag{5.6}$$

By (5.5) and (5.6) we rewrite (5.1) in the form

$$p(t) + \sum_{n=1}^{\infty} L_n(t, \lambda) \int_0^T G_n(s, \lambda) f\big[s, \varphi\big(s, p(s), \beta\big)\big]ds = \sum_{n=1}^{\infty} L_n(t, \lambda) h_n, \tag{5.7}$$

or in the operator form

$$p(t) = G\big[p(t)\big], \tag{5.8}$$

where

$$G\big[p(t)\big] = \sum_{n=1}^{\infty} L_n(t, \lambda) \left[h_n - \int_0^T G_n(s, \lambda) f\big[s, \varphi\big(s, p(s), \beta\big)\big]ds\right]. \tag{5.9}$$

Now we turn to the problem of unique solvability of operator equation (5.8).

Lemma 5.2 *The operator G maps the space $H(0, T)$ into itself, i.e. $G[p(t)]$ is an element of the space $H(0, T)$.*

Proof By direct calculation we have the inequality

$$
\begin{aligned}
&\int_0^T G^2\big[p(t)\big]dt \\
&\quad = \int_0^T \left(\sum_{n=1}^{\infty} L_n(t,\lambda)\left[h_n - \int_0^T G_n(s,\lambda) f\big[s, \varphi\big(s, p(s), \beta\big)\big]ds \right] \right)^2 dt \\
&\quad \le 2\int_0^T \sum_{n=1}^{\infty} L_n^2(t,\lambda) \sum_{n=1}^{\infty} \left[h_n^2 + \int_0^T G_n^2(s,\lambda)ds \int_0^T f^2\big[s, \varphi\big(s, p(s), \beta\big)\big]ds \right] dt \\
&\quad \le 2\int_0^T \sum_{n=1}^{\infty} 2g_n^2(t) \Bigg[e^{-2\lambda_n^2(T-t)} \\
&\qquad + \lambda^2 \int_0^T P_n^2(\tau, t, \lambda)d\tau \int_0^T e^{-2\lambda_n^2(T-\tau)}d\tau \Bigg] \Bigg\{ 2\Bigg[\big\|\xi(x)\big\|_H^2 \\
&\qquad + 2\left(1 + \frac{\lambda^2 K_0}{\big(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T}\big)^2} \frac{1}{2\lambda_1^2}\right) \big\|\psi(x)\big\|_H^2 \Bigg] \\
&\qquad + \sum_{n=1}^{\infty} 2g_n^2(t) \left(1 + \frac{\lambda^2 K_0 T}{\big(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0 T}\big)^2}\right) \big\| f\big[s, \varphi\big(s, p(s), \beta\big)\big] \big\|_H^2 \Bigg\} dt \\
&\quad \le C \int_0^T \sum_{n=1}^{\infty} g_n^2(t)dt < \infty,
\end{aligned}
$$

from which the statement of the lemma follows. □

Lemma 5.3 *Suppose conditions*

$$\big\| f\big[t, u(t)\big] - f\big[t, \overline{u}(t)\big] \big\|_H \le f_0 \big\| u(t) - \overline{u}(t) \big\|_H, \quad f_0 > 0 \tag{5.10}$$

and

$$\big\| \varphi\big[t, p(t), \beta\big] - \varphi\big[t, \overline{p}(t), \beta\big] \big\|_H \le \varphi_0(\beta) \big\| p(t) - \overline{p}(t) \big\|_H, \quad \varphi_0(\beta) > 0 \tag{5.11}$$

are satisfied. Then if the condition

$$\gamma = 2\big\| g(t,x) \big\|_H^2 \left(1 + \frac{a_0^2 K_0}{\big(\sqrt{2\lambda_1^2} - a_0\sqrt{K_0 T}\big)^2}\right) f_0 \varphi_0(\beta) < 1, \tag{5.12}$$

is met, where α_0 is a positive constant satisfying the inequality

$$|\lambda| \le a_0 < \frac{\sqrt{2}\lambda_1}{\sqrt{K_0 T}}, \tag{5.13}$$

then the operator G is contractive.

Proof By direct calculations, we have the inequality

$$\int_0^T \left|G[p]-G[\overline{p}]\right|^2 dt$$

$$=\int_0^T \left(\sum_{n=1}^{\infty} L_n(t,\lambda)\int_0^T G_n(s,\lambda)\right.$$

$$\left.\times \left(f\left[s,\varphi\left(s,p(s),\beta\right)\right]-f\left[s,\varphi\left(s,\overline{p}(s),\beta\right)\right]\right)ds\right)^2 dt$$

$$\le \int_0^T \sum_{n=1}^{\infty} L_n^2(t,\lambda)\sum_{n=1}^{\infty}\int_0^T G_n^2(s,\lambda)ds\int_0^T \left(f\left[s,\varphi\left(s,p(s),\beta\right)\right]\right.$$

$$\left.-f\left[s,\varphi\left(s,\overline{p}(s),\beta\right)\right]\right)^2 ds dt$$

$$\le \left[2\left\|g(t,x)\right\|_H^2\left(1+\frac{\lambda^2 K_0}{(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T})^2}\right)f_0\varphi_0(\beta)\left\|p(s)-\overline{p}(s)\right\|_H\right]^2,$$

from which we find that

$$\left\|G[p]-G[\overline{p}]\right\|_H$$

$$\le 2\left\|g(t,x)\right\|_H^2\left(1+\frac{\lambda^2 K_0}{(\sqrt{2\lambda_1^2}-|\lambda|\sqrt{K_0 T})^2}\right)f_0\varphi_0(\beta)\left\|p(t)-\overline{p}(t)\right\|_H.$$

□

Theorem 5.4 *Suppose that conditions* (2.4)–(2.5), (3.6), (4.5), (5.10)–(5.13) *are satisfied. Then operator equation* (5.8) *has a unique solution in the space* $H(0,T)$.

Proof According to Lemmas 5.1 and 5.2, operator equation (5.8) can be considered in the space $H(0,T)$. According to Lemma 5.3 operator G is contractive. Since the Hilbert space $H(0,T)$ is a complete metric space, by the theorem on contraction mappings the operator G has a unique fixed point, i.e. operator equation (5.8) has a unique solution. □

The solution of operator equation (5.8) can be found by the method of successive approximations, i.e. nth approximation of the solution is found by the formula

$$p_n(t)=G\left[p_{n-1}(t)\right], \quad n=1,2,3,\dots,$$

where $p_0(t)$ is an arbitary element of the space $H(0,T)$, and we have the estimate

$$\left\|\overline{p}(t)-p_n(t)\right\|\le \frac{\gamma^n}{1-\gamma}\left\|G\left[p_0(t)\right]-p_0(t)\right\|_H, \tag{5.14}$$

where $0 < \gamma < 1$ is the construction constant. The exact solution can be found as the limit of the approximate solutions, i.e.

$$\overline{p}(t) = \lim_{n\to\infty} p_n(t).$$

Substituting this solution in (5.6) we find the required optimal control

$$u^0(t) = \varphi\big[t, \overline{p}(t), \beta\big]. \tag{5.15}$$

The optimal process $v^0(t,x)$, i.e. the solution of boundary value problem (2.1)–(2.5) corresponding to the optimal control $u^0(t)$, according to (2.6) and (2.7) we find from the formula

$$\begin{aligned} v^0(t,x) &= \sum_{n=1}^{\infty}\left(\lambda\int_0^T R_n(t,s,\lambda)a_n(s)ds - a_n(t)\right)z_n(x) \\ &= \sum_{n=1}^{\infty}\Bigg[\psi_n\left(e^{-\lambda_n^2 t} + \lambda\int_0^T R_n(t,s,\lambda)e^{-\lambda_n^2 s}\right)ds \\ &\quad + \int_0^T e^{-\lambda_n^2(t-\tau)}g_n(\tau)f\big[\tau, u^0(\tau)\big]d\tau \\ &\quad + \lambda\int_0^T R_n(t,s,\lambda)\int_0^s e^{-\lambda_n^2(s-\eta)}g_n(\eta)f\big[\eta, u^0(\eta)\big]d\eta ds\Bigg]z_n(x). \end{aligned} \tag{5.16}$$

The minimum value of the functional (3.2) is calculated by the formula

$$J\big[u^0(t)\big] = \int_0^1\big[v^0(T,x) - \xi(x)\big]^2dx + \beta\int_0^T M\big[t, u^0(t)\big]dt. \tag{5.17}$$

The found triple $(u^0(t), v^0(t,x), J[u^0(t)])$ is a solution of the nonlinear optimization problem.

6 An Approximate Solution of the Optimization Problem

In practice, it is not always possible to find the exact solution of (5.8), i.e. the limit function $\overline{p}(t)$. Therefore, in most of the cases only approximate solutions $p_k(t)$ of (5.8) are looked for, where the number k is determined by the inequality

$$\big\|\overline{p}(t) - p_k(t)\big\|_H \le \frac{\gamma^k}{1-\gamma}\big\|G\big[p_0(t)\big] - p_0(t)\big\|_H < \varepsilon \tag{6.1}$$

for given $\varepsilon > 0$. By substituting the approximate solution $p_k(t)$ in (5.6) we find the kth approximation of optimal control

$$u_k(t) = \varphi\big[t, p_k(t), \beta\big]. \tag{6.2}$$

Lemma 6.1 *Let the function $\varphi[t, \vartheta(t), \beta]$ satisfy the Lipschitz condition with respect to the functional variable $\vartheta(t)$, i.e.*

$$\begin{aligned} &\big\|\varphi\big[t, \vartheta_1(t), \beta\big] - \varphi\big[t, \vartheta_2(t), \beta\big]\big\|_H \le \varphi_0(\beta)\big\|\big(\vartheta_1(t) - \vartheta_2(t)\big\|\big)_H, \\ &\varphi_0(\beta) > 0. \end{aligned} \tag{6.3}$$

Then the kth approximate controls converge to the optimal control $u^0(t)$ in the norm of the Hilbert space $H(Q)$ as $k \to \infty$.

Proof Lemma's assertion follows from the inequality

$$\begin{aligned} \big\|u^0(t) - u_k(t)\big\|_H &= \big\|\varphi\big[t, p^0(t), \beta\big] - \varphi\big[t, p_k(t), \beta\big]\big\|_H \\ &\le \varphi_0(\beta)\big\|p^0(t) - p_k(t)\big\|_H \\ &\le \varphi_0(\beta)\frac{\gamma^k}{1-\gamma}\big\|G\big[p_0(t)\big] - p_0(t)\big\|_H \xrightarrow[k\to\infty]{} 0. \end{aligned} \tag{6.4}$$

□

Lemma 6.2 *Let the function $f[t, u(t)]$ satisfy the Lipschitz condition with respect to the functional variable $u(t)$, i.e.*

$$\big\|f\big[t, u_1(t)\big] - f\big[t, u_2(t)\big]\big\|_H \le f_0\big\|u_1(t) - u_2(t)\big\|_H \tag{6.5}$$

and we have (6.3). *Then m, kth approximations of the solution $v_k^m(t, x)$ of boundary value problem* (2.1)–(2.3) *converge to the exact solution $v(t, x)$ in the norm of the Hilbert space $H(Q)$ as $m, k \to \infty$.*

Proof Approximations of the optimal process $v^0(t, x)$ are determined by two indices k and m and have the form

$$\begin{aligned} v_k^m(t, x) = \sum_{n=1}^{\infty}\Bigg\{&\psi_n\bigg(e^{-\lambda_n^2 t} + \lambda\int_0^T R_n^m(t, s, \lambda)e^{-\lambda_n^2 s}ds\bigg) \\ &+ \int_0^t e^{-\lambda_n^2(t-\tau)}g_n(\tau)f\big[\tau, u_k(\tau)\big]d\tau \\ &+ \lambda\int_0^T g_n(\tau)\int_0^T R_n^m(t, s, \lambda)e^{-\lambda_n^2(s-\tau)}ds f\big[\tau, u_k(\tau)\big]d\tau\Bigg\}z_n(x). \end{aligned} \tag{6.6}$$

Since

$$\begin{aligned} &v^0(t, x) - v_k^m(t, x) \\ &\quad = \sum_{n=1}^{\infty}\bigg\{\psi_n\lambda\int_0^T\big[R_n(t, s, \lambda) - R_n^m(t, s, \lambda)\big]e^{-\lambda_n^2 s}\bigg\}ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^T e^{-\lambda_n^2(t-\tau)} g_n(\tau)\bigl[f\bigl(\tau, u^0(\tau)\bigr) - f\bigl(\tau, u_k(\tau)\bigr)\bigr]d\tau \\
&+\lambda \int_0^T g_n(\tau) \int_\tau^T \bigl[R_n(t,s,\lambda) - R_n^m(t,s,\lambda)\bigr]e^{-\lambda_n^2(s-\tau)} ds f\bigl[\tau, u^0(\tau)\bigr]d\tau \\
&+\lambda \int_0^T g_n(\tau) \int_\tau^T R_n^m(t,s,\lambda)e^{-\lambda_n^2(s-\tau)} ds \\
&\times \bigl[f\bigl(\tau, u^0(\tau)\bigr) - f\bigl(\tau, u_k(\tau)\bigr)\bigr]d\tau z_n(x),
\end{aligned}
$$

by calculations, used to prove the convergence of the approximate solutions of boundary value problem (2.1)–(2.3), we get the relation

$$
\bigl\|v^0(t,x) - v_k^m(t,x)\bigr\|_H^2 \le C_1(\lambda)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m} + C_2(\lambda)\left(\frac{\gamma^k}{1-\gamma}\right)^2 \xrightarrow[m,k\to\infty]{} 0,
$$

where

$$
C_1(\lambda) = \bigl(\bigl\|\psi(x)\bigr\|_H^2 + \bigl\|f\bigl[t, u(t)\bigr]\bigr\|_H^2 \cdot \bigl\|g(t,x)\bigr\|_H^2\bigr)\frac{\lambda^2 K_0 T}{4}\left(1 - \frac{1}{\ln|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}}\right),
$$

$$
\begin{aligned}
C_2(\lambda) = 4&\left(1 + \frac{\lambda^2 K_0}{2\lambda_1^2(\sqrt{2\lambda_1^2} - |\lambda|\sqrt{K_0T})^2}\right) \\
&\times T\bigl\|g(t,x)\bigr\|_H^2 f_0^2\varphi_0^2(\beta)\bigl\|G\bigl[p_0(t)\bigr] - p_0(t)\bigr\|_H^2,
\end{aligned}
$$

by which assertion of the lemma follows. □

Lemma 6.3 *The m,kth approximations $J_m[u_k(t)]$ of the minimum value of the functional $J[u^0(t)]$ converges to the exact value as $m,k \to \infty$.*

Proof Since

$$
J\bigl[u^0(t)\bigr] = \int_Q \bigl[v^0(T,x) - \xi(x)\bigr]^2 dx + \beta\int_0^T M\bigl[t, u^0(t)\bigr]dt,
$$

$$
J_m\bigl[u_k(t)\bigr] = \int_Q \bigl[v_k^m(T,x) - \xi(x)\bigr]^2 dx + \beta\int_0^T M\bigl[t, u_k(t)\bigr]dt,
$$

it is not difficult to obtain the inequality

$$
\begin{aligned}
&\bigl|J\bigl[u^0(t)\bigr] - J_m\bigl[u_k(t)\bigr]\bigr| \\
&\quad = \int_Q \bigl\{\bigl[v^0(T,x) - \xi(x)\bigr]^2 - \bigl[v_k^m(T,x) - \xi(x)\bigr]^2\bigr\}dx \\
&\qquad + 2\beta\int_0^T \bigl(M\bigl[t, u^0(t)\bigr] - M\bigl[t, u_k(t)\bigr]\bigr)dt
\end{aligned}
$$

$$\begin{aligned}
&\le \left\|v^0(T,x)+v_k^m(T,x)-2\xi(x)\right\|_H \left\|v^0(T,x)-v_k^m(T,x)\right\|_H \\
&\quad + 2\beta\sqrt{T}\left\|M\left[t,u^0(t)\right]-M\left[t,u_k(t)\right]\right\|_H \\
&\le \left\|v^0(T,x)+v_k^m(T,x)-2\xi(x)\right\|_H \left\|v^0(T,x)-v_k^m(T,x)\right\|_H \\
&\quad + 2\beta\sqrt{T}m_0\left\|u^0(t)-u_k(t)\right\|_H.
\end{aligned}$$

By Lemmas 6.1 and 6.2, and in view of the fact that

$$\begin{aligned}
&u^0(t)\in H(0,T), \qquad v(T,x)\in H(0,1),\\
&\xi(x)\in H(0,1), \qquad f\left[t,u(t)\right]\in H(0,T),
\end{aligned}$$

we obtain the relation

$$\begin{aligned}
\left|J\left[u^0(t)\right]-J_m\left[u_k(t)\right]\right| &\le C_0(\lambda)\left[C_1(\lambda)\left(|\lambda|\sqrt{\frac{K_0T}{2\lambda_1^2}}\right)^{2m}+C_2(\lambda)\left(\frac{\gamma^k}{1-\gamma}\right)^2\right]^{1/2}\\
&\quad + 2\beta\sqrt{T}\varphi_0(\beta)\left\|G\left[p_0(t)\right]-p_0(t)\right\|_H\frac{\gamma^k}{1-\gamma}\xrightarrow[m,k\to\infty]{}0,
\end{aligned}$$

where

$$C_0(\lambda)\ge\left\|v^0(T,x)+v_m^k(T,x)-2\xi(x)\right\|_H,$$

by which the statement of the lemma follows. □

Thus, by Lemmas 6.1–6.3 it is proved that the approximate solutions $(u^k(t), v_m^k(t,x), J[u^k(t)])$ of the problem of the nonlinear optimization converge to the exact solution $(u^0(t), v^0(t,x), J[u^0(t)])$ with respect to control, optimal process and functional.

7 Conclusion

The obtained results are theoretical and can be used to develop methods for studying optimal control of systems with nonlinear by distributed parameter and constructive methods for solving them. They can be applied to solving applied problems.

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